Resit Exam — Analysis (WBMA012-05)

Thursday 14 April 2022, 8.30h–10.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (5 + 8 + 8 = 21 points)

Assume that $A \subseteq \mathbb{R}$ is nonempty, and define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sup\{1 - |x - a| \,:\, a \in A\}.$$

(a) Use the Axiom of Completeness to explain why the supremum exists.

(b) Prove that if $x \in \overline{A}$, then $f(x) = 1$.

(c) Compute $f(x)$ when $A = \mathbb{Q}$ and $x = \sqrt{2}$. Motivate your answer.

Problem 2 (8 + 8 + 8 = 24 points)

You may assume without proof that $\pi = 3.14\ldots$ is irrational and that $x = 1$ is a limit point of the set

$$A = \{\cos(n) \,:\, n \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of all natural numbers.

Show that A is *not* compact in three different ways:

(a) A does not satisfy the definition of a compact set;

(b) A is not closed;

(c) A has an open cover without a finite subcover.

Please turn over for problems 3 and 4...

Problem 3 ($8 + [3 + 10] = 21$ points)

(a) Formulate the Intermediate Value Theorem and the Mean Value Theorem.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(1) = 1$. Prove the following statements:

(i) There exists a point $0 < b < 1$ such that $f(b) = 1/2$.

(ii) There exist points $0 < c_1 < c_2 < 1$ such that

$$\frac{1}{f'(c_1)} + \frac{1}{f'(c_2)} = 2.$$

Hint: consider the intervals $[0, b]$ and $[b, 1]$.

Problem 4 ($4 + 4 + 8 + 8 = 24$ points)

(a) Let $x \neq 0$. Show that

$$1 + e^x + e^{2x} + \cdots + e^{(n-1)x} = \frac{1}{1 - e^x} - \frac{e^{nx}}{1 - e^x} \quad \text{for all} \quad n \in \mathbb{N}.$$

(b) Prove that the infinite series $\sum_{k=0}^{\infty} e^{kx}$ converges pointwise on the interval $(-\infty, 0)$.

(c) Prove that for all $a < 0$ the convergence is uniform on the interval $(-\infty, a)$.

(d) Is the convergence still uniform on the interval $(-\infty, 0)$?

End of test (90 points)

Solution of problem 1 (5 + 8 + 8 = 21 points)

(a) Let $x \in \mathbb{R}$ be arbitrary. The set $\{1 - |x - a| \,:\, a \in A\}$ is nonempty since A is nonempty.
(1 point)

In addition, for any $a \in A$ we have that $|x - a| \geq 0$ which implies that $1 - |x - a| \leq 1$. This shows that the set $\{1 - |x - a| \,:\, a \in A\}$ is bounded above. By the Axiom of Completeness it follows that the least upper bound of this set exists.
(4 points)

(b) Recall that $\overline{A} = A \cup L$, where L is the set of all limit points of A. Assume that $x \in \overline{A}$. Then we must distinguish between two cases.

If $x \in A$, then with $a = x$ we have $1 - |x - a| = 1$ is the largest element of the set $\{1 - |x - a| \,:\, a \in A\}$. In this case, the supremum is just a maximum and thus $f(x) = 1$.
(2 points)

If $x \notin A$, then x must be a limit point of A. There are at least two different ways to show that $f(x) = 1$.

Method 1. For all $\epsilon > 0$ there exists $a_\epsilon \in A$ such that $0 < |x - a_\epsilon| < \epsilon$, which implies that $1 - |x - a_\epsilon| > 1 - \epsilon$. This shows that no number $u < 1$ can be an upper bound for the set $\{1 - |x - a| \,:\, a \in A\}$. Since in part (a) it has been shown that $u = 1$ is an upper bound, it follows that $u = 1$ is the least upper bound.
(6 points)

Method 2. There exists a sequence (a_n) in A such that $\lim a_n = x$ and $a_n \neq x$ for all $n \in \mathbb{N}$. Assume that u is an upper bound for the set $\{1 - |x - a| \,:\, a \in A\}$. We need to show that $u \geq 1$. Note that $1 - |x - a_n| \leq u$ for all $n \in \mathbb{N}$. Since $\lim |x - a_n| = 0$ it follows by the Order Limit Theorem that $1 \leq u$.
(6 points)

(c) *Method 1.* In the lectures it has been proven that $\overline{\mathbb{Q}} = \mathbb{R}$. Since $\sqrt{2} \in \mathbb{R}$, it follows immediately by part (b) that $f(x) = 1$.
(8 points)

Method 2. The rational numbers are dense in the real numbers, which means that for any real numbers s and t with $s < t$ there exists a rational number r such that $s < r < t$.
(2 points)

In particular, for any $\epsilon > 0$ there exists $a \in A = \mathbb{Q}$ such that $\sqrt{2} < a < \sqrt{2} + \epsilon$. This implies that $1 - |\sqrt{2} - a| > 1 - \epsilon$.
(3 points)

Hence, no number $u < 1$ can be an upper bound of the set $\{1 - |\sqrt{2} - a| \,:\, a \in \mathbb{Q}\}$. Since 1 is an upper bound, we conclude that $f(\sqrt{2}) = 1$.
(3 points)

Solution of problem 2 (8 + 8 + 8 = 24 points)

(a) Since it is given that $x = 1$ is a limit point of A, it follows that there exists a sequence (a_n) in A such that $a_n \neq x$ for all $n \in \mathbb{N}$ and $\lim a_n = x$.
(3 points)

Any subsequence (a_{n_k}) of (a_n) is then also convergent and has limit x.
(1 point)

However, $x \notin A$. Indeed, if $\cos(n) = 1$ for some $n \in \mathbb{N}$, then $n = 2k\pi$ for some positive integer k. This would imply that $\pi = n/2k$ is rational which is a contradiction.
(2 points)

We conclude that there exists at least one sequence (a_n) in A for which all subsequences have a limit which is not contained in A. This shows that A is not compact.
(2 points)

(b) It is given that $x = 1$ is a limit point of A. However, as in part (a) we can argue that $x \notin A$. This means that A does not contain all its limit points.
(2 points)

In order for the set A to be closed, it must contain all its limit points which is not the case. Hence, A is not closed.
(3 points)

If the set A is compact, then A is both closed and bounded. However, we know that A is not closed and therefore also not compact.
(3 points)

(c) As in part (a) we can argue that $x \notin A$. This implies that $\cos(n) < 1$ for all $n \in \mathbb{N}$. Let $O_\lambda = (-2, \lambda)$, where $\lambda \in \Lambda = (0, 1)$. Then

$$A \subseteq (-2, 1) = \bigcup_{\lambda \in \Lambda} O_\lambda$$

which means that the sets O_λ form an open cover for A. (Note: there are many possibilities for choosing the open sets O_λ and the index set Λ.)
(2 points)

Assume that $A \subseteq O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}$ for some finite choice of indices $\{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda$. Let $M = \max\{\lambda_1, \ldots, \lambda_n\}$, then $M < 1$ and $A \subseteq (-2, M)$.
(2 points)

Recall $x = 1$ is a limit point of A. This implies that for all $\epsilon > 0$ there exists $a \in A$ such that $0 < |a - 1| < \epsilon$, or, equivalently, $a \in (1 - \epsilon, 1)$ since $a < 1$. Taking $\epsilon < 1 - M$ gives $M < a$, which contradicts that $A \subseteq (-2, M)$. We conclude that the open cover O_λ for A does not have a finite subcover. Therefore, A is not compact.
(4 points)

Solution of problem 3 (8 + [3 + 10] = 21 points)

(a) *Intermediate Value Theorem:* Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.
(4 points)

Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(4 points)

(b) (i) Since $f : [0, 1] \to \mathbb{R}$ is differentiable it is also continuous.
(1 point)

Applying the Intermediate Value Theorem to f on the interval $[0, 1]$ with $L = 1/2$ gives the existence of a point $b \in (0, 1)$ such that $f(b) = 1/2$.
(2 points)

(ii) Applying the Intermediate Value Theorem to f on the interval $[0, b]$ gives the existence of a point $c_1 \in (0, b)$ such that

$$f'(c_1) = \frac{f(b) - f(0)}{b - 0} = \frac{1}{2b}.$$

(4 points)

Applying the Intermediate Value Theorem to f on the interval $[b, 1]$ gives the existence of a point $c_2 \in (b, 1)$ such that

$$f'(c_2) = \frac{f(1) - f(b)}{1 - b} = \frac{1}{2(1 - b)}.$$

(4 points)

Finally, we obtain

$$\frac{1}{f'(c_1)} + \frac{1}{f'(c_2)} = 2b + 2(1 - b) = 2.$$

(2 points)

Solution of problem 4 (4 + 4 + 8 + 8 = 24 points)

(a) *Method 1.* An elementary computation shows that

$$(1 + e^x + e^{2x} + \cdots + e^{(n-1)x})(1 - e^x) = 1 - e^{nx}.$$

Dividing by $1 - e^x$ on both sides and rearranging terms gives the desired result.
(4 points)

Method 2. For $n = 1$ we have

$$\frac{1}{1-e^x} - \frac{e^{nx}}{1-e^x} = \frac{1-e^x}{1-e^x} = 1,$$

which shows that the requested formula holds for $n = 1$.
(1 point)

Now assume that the formula holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} 1 + e^x + e^{2x} + \cdots + e^{(n-1)x} + e^{nx} &= \frac{1}{1-e^x} - \frac{e^{nx}}{1-e^x} + e^{nx} \\ &= \frac{1}{1-e^x} - \frac{e^{nx}}{1-e^x} + \frac{e^{nx} - e^{(n+1)x}}{1-e^x} \\ &= \frac{1}{1-e^x} - \frac{e^{(n+1)x}}{1-e^x}, \end{aligned}$$

which shows that the formula also holds for $n + 1$. By induction, the formula holds for all $n \in \mathbb{N}$.
(3 points)

(b) *Method 1.* Consider the partial sums

$$s_n(x) = 1 + e^x + e^{2x} + \cdots + e^{(n-1)x}.$$

For fixed $x \in (-\infty, 0)$ we have that $e^x \in (0, 1)$ and thus $\lim e^{nx} = \lim(e^x)^n = 0$. Using the Algebraic Limit Theorem gives

$$\lim s_n(x) = \lim\left(\frac{1}{1-e^x} - \frac{e^{(n+1)x}}{1-e^x}\right) = \frac{1}{1-e^x} - \frac{1}{1-e^x}\lim e^{nx} = \frac{1}{1-e^x}.$$

(4 points)

Method 2. Consider the partial sums

$$s_n(x) = 1 + e^x + e^{2x} + \cdots + e^{(n-1)x}.$$

For fixed $x \in (-\infty, 0)$ we have that $e^x \in (0, 1)$ and thus $\lim e^{nx} = \lim(e^x)^n = 0$. This means that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |e^{nx} - 0| < (1 - e^x)\epsilon.$$

Therefore,

$$n \geq N \quad \Rightarrow \quad |s_n(x) - f(x)| = \frac{|e^{nx} - 0|}{1 - e^x} < \epsilon.$$

(4 points)

(c) *Method 1.* If $x \in (-\infty, a)$, then $e^{nx} < e^{na}$ and $1 - e^x > 1 - e^a$. This implies that

$$|s_n(x) - f(x)| < \frac{(e^a)^n}{1 - e^a},$$

and thus

$$\sup_{x \in (-\infty, a)} |s_n(x) - f(x)| \leq \frac{e^{na}}{1 - e^a}.$$

(4 points)

Since $a < 0$ it follows that $\lim e^{na} = 0$. By the Order Limit Theorem it then follows that

$$\lim \left(\sup_{x \in (-\infty, a)} |s_n(x) - f(x)| \right) = 0,$$

which implies that the sequence (s_n) converges uniformly to f on $(-\infty, a)$.
(4 points)

Method 2. For all $x \in (-\infty, a)$ we have

$$|s_n(x) - f(x)| < \frac{e^{na}}{1 - e^a},$$

Let $\epsilon > 0$ be arbitrary and take $N \in \mathbb{N}$ such that $N > \ln((1 - e^a)\epsilon)/a$. Then

$$n \geq N \quad \Rightarrow \quad |s_n(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in (-\infty, a),$$

which means by definition that the sequence (s_n) converges uniformly to f on $(-\infty, a)$.
(8 points)

(d) For any fixed $n \in \mathbb{N}$ we have

$$|s_n(x) - f(x)| = \frac{e^{nx}}{1 - e^x},$$

and the right hand side is unbounded on the interval $(-\infty, 0)$.
(4 points)

In particular, it is *not* true that

$$\lim \left(\sup_{x \in (-\infty, a)} |s_n(x) - f(x)| \right) = 0,$$

which implies that the sequence (s_n) does not converge uniformly to f on $(-\infty, 0)$.
(4 points)